Uniform second-order solution for supersonic flow over delta wing using reverse-flow integral method

By JOSEPH H. CLARKE AND JAMES WALLACE
Division of Engineering, Brown University, Providence, Rhode Island

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The problem of supersonic flow over an inclined flat-plate delta wing with supersonic edges is solved to second order in incidence. This solution for surface pressure is uniform and fully analytic. The approach utilizes a reverse-flow integral method previously developed for second-order problems. This method is augmented by a number of techniques appropriate to its framework. The simplification over standard techniques achieved by using these reverse-flow methods is quite substantial and makes the problem tractable.

Reverse-flow procedures give a volume-surface integral relation that connects the second-order forward flow over the body of interest with the linearized reverse-flow over a related body. A singular integral equation is generated from the integral relation by introducing the edge sweep of the reverse-flow wing as a free parameter. An inversion is available which gives the second-order solution on the surface of the wing. The solution is then made uniformly valid using techniques previously developed.

1. Introduction

Using reverse-flow integral methods, this paper presents an analytic, uniform second-order solution for the surface pressure on a flat-plate delta wing with supersonic edges and small incidence. These second-order solutions are particularly relevant (see Clarke 1962) for flight Mach numbers between the linear range and the hypersonic range. The reverse-flow method for evaluating aerodynamic forces or solving for the surface pressure distribution to second-order was developed by Clarke (1963). It exploits integral relations between the flow over a body of interest and the flow over a related body in which the velocity at infinity is equal in magnitude but opposite in direction. Parts of the formalism and results can be traced for their origin to the reverse-flow relations of linearized theory (see Ward 1955). The present research represents the first application in obtaining the surface pressure and, at the same time, makes several additional theoretical and practical contributions to the reverse-flow method. The specific results of the present work can be generalized readily to other types of conical flow fields.

The problem of non-linear supersonic flow over a delta wing continues to be of great interest because it represents a fundamental configuration which has been considered frequently but not really solved in this range by any method,† to the writers' knowledge. It consequently provides an interesting test of the usefulness of the integral method for determining higher approximations to the surface pressure. The governing differential equation of second-order theory is a three-dimensional, linear, inhomogeneous wave equation and a formal solution may be written in terms of volume and surface integrals of the singular source function. The volume integral is especially difficult to calculate at the variable field point because the sources are distributed over the spatial region bounded by the downstream Mach cone from the vertex. The intensity of this source distribution is a complicated function of the linearized solution. But the simplification achieved with reverse-flow methods is quite substantial and tends to make the problem tractable.

The method begins with a volume-surface integral identity that connects the second-order forward flow field over the delta wing with the reverse flow field over a related body. Further this reverse flow may be taken as the linearized flow over a wing with an arbitrary sweep parameter. The volume integral extends over the region bounded by the wing and the envelopes of disturbance of the forward and reverse flow; it contains only known first-order quantities from the forward and reverse flow fields and can be evaluated. The integrals over the body surface contain the second-order quantities and, after use of the tangency condition, only one unknown second-order quantity remains, namely, the essential second-order part of the surface pressure. This integral relation in the last named quantity is identified as a singular integral equation whose kernel contains the reverse-flow sweep parameter, the current variable of the equation. An inversion, obtained by the method of Stieltjes's transform, is available; under the inversion the sweep parameter takes on the significance of the conical variable in the plane of the delta wing. The method of solution utilizes certain properties of a surface integral of first-order quantities over the envelope of disturbance in the reverse-flow. Because of the singular behaviour of this surface integral, an arbitrary analytic function is introduced into the solution, and this must be determined by examination of the behaviour of the solution near the Mach cone. After determination of the inversion, the resulting solution is made uniform. It may be continued off the plane of the wing and may also be generalized to delta wings with a specified distribution of incidence.

2. Aspects of first- and second-order theory

Consider the steady supersonic homentropic flow of a perfect compressible gas past a quasi-cylindrical body. If Φ denotes the exact perturbation potential divided by U and U is the free-stream speed in the direction of the Cartesian co-ordinate x, then the exact equation for Φ is

$$(2a^2/U^2)\,\nabla^2\Phi = [\nabla(x+\Phi)\,.\,\nabla]\,[\nabla(x+\Phi)\,.\,\nabla(x+\Phi)], \tag{1}$$

† [*Note added in proof.*] An exact solution for flow over a delta wing has recently been obtained using an elaborate numerical iteration procedure. The present first author, in his review subjoining the translation, offers some comments on this numerical solution and the present analytic solution. See Babaev, D. A. 1963 *AIAA J.* **1**, 2224, Russian Supplement.

where $$a^2/U^2 = 1/M^2 - \tfrac{1}{2}(\gamma-1)\,(2\Phi_x + \Phi_x^2 + \Phi_y^2 + \Phi_z^2), \tag{2}$$

M is the free-stream Mach number, and γ is the specific-heat ratio. In supersonic flow, the potential Φ and its derivatives vanish at all points upstream of the shock or characteristic surface which bounds the influence domain of the body. On shock surfaces with normal $\mathbf{n}$, the tangential velocity component must be continuous; then
$$\Delta(\mathbf{n} \times \nabla\Phi) = 0 \quad \text{or} \quad \Delta\Phi = 0. \tag{3}$$

The symbol Δ signifies the jump in the quantity across the shock. The jump in the normal component of the perturbation velocity is given by
$$\Delta(\mathbf{n}\,.\,\nabla\Phi) = \frac{2}{\gamma+1}\left[\frac{a^2/U^2}{\mathbf{n}\,.\,\nabla(x+\Phi)} - \mathbf{n}\,.\,\nabla(x+\Phi)\right]. \tag{4}$$

The quantities on the right-hand side of (4) are evaluated on the upstream side of the shock surfaces. In addition the normal derivative of $(x+\Phi)$ must vanish on solid bodies; then
$$\mathbf{n}\,.\,\nabla\Phi = -\mathbf{n}\,.\,\mathbf{i} \quad \text{on} \quad z = Z(x,y), \tag{5}$$

with $Z(x,y)$ proportional to the small parameter α.

Second-order theory represents the second step in an expansion or iteration procedure in terms of the small parameter α. For quasi-cyclindrical bodies such as the delta wing, the ordinary representation of the potential is of the form
$$\Phi = \varphi + f + O(\alpha^3), \quad \varphi = O(\alpha), \quad f = O(\alpha^2). \tag{6}$$

By substituting (6) into (1) under the assumption $\alpha^2 \ll 1$, one obtains the following equations
$$-B^2\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0, \tag{7}$$
$$-B^2 f_{xx} + f_{yy} + f_{zz} = 2M^2[(N-1)\,B^2\varphi_x\varphi_{xx} + \varphi_y\varphi_{xy} + \varphi_z\varphi_{zx}], \tag{8}$$
where $$N = (\gamma+1)\,M^2/2B^2. \tag{9}$$

The tangency condition (5) becomes, after expanding in a Maclaurin series about $z = 0$,
$$\varphi_z = Z_x(x,y), \quad f_z = Z_x(x,y)\,\varphi_x + Z_y(x,y)\,\varphi_y - Z(x,y)\,\varphi_{zz}, \quad \text{on} \quad z = 0. \tag{10a,b}$$

For a flat-plate delta wing we have $Z_x(x,y) = \alpha$, $Z_y(x,y) = 0$ and (10) becomes
$$\varphi_z = \alpha, \quad f_z = \alpha\varphi_x - \alpha x\varphi_{zz}, \quad \text{on} \quad z = 0. \tag{11a,b}$$

No explicit use can be made of the shock conditions (3) and (4) when the ordinary expansion (6) is assumed. The pressure coefficient is given by
$$c_p = -2(\varphi_x + f_x) - \varphi_y^2 - \varphi_z^2 + B^2\varphi_x^2. \tag{12}$$

This approximation to the potential is non-uniform and fails near the characteristic surfaces where φ_{xx}, φ_{yx} and φ_{zx} are singular. Furthermore, the tangency condition (10) is incorrect whenever the characteristic surfaces interact the wing surface unless $Z(x,y) = 0$ at these points. Modification of the expansion process (6) to remove these difficulties was first undertaken by Lighthill (1949), who proposed expansion of both the dependent variable Φ and the independent variable x in a series in α and a new variable u. Then the approximation to the potential takes the form

$$\Phi = \varphi(u,y,z)+f(u,y,z)+O(\alpha^3), \tag{13}$$

$$x = u+x_1(u,y,z)+x_2(u,y,z)+O(\alpha^3), \quad x_1 = O(\alpha), \quad x_2 = O(\alpha^2). \tag{14}$$

We have first to determine x_1 by requiring the second-order solution to exhibit the correct behaviour, and then x_2 by requiring the third-order solution to exhibit the correct behaviour. Further, Wallace & Clarke (1963) have suggested that any ordinary second-order solution can be made uniformly valid by application of an extended version of Lighthill's (1954) principle. This extended principle states that the second-order flow field is made uniformly valid to second order if distance downstream from the foremost envelope of disturbances is always reinterpreted therein as distance downstream from the limit surfaces, providing the term $x_1(x,y,z)\,\varphi_x(x,y,z)$ is first added to the ordinary second-order potential. The types of failure in the ordinary solution depend upon whether the wave is like that due to a wedge or to a cone in nature. The function x_1 is then determined to insure that either (a) for cone-like waves, the solution does not become increasingly singular, or (b) for wedge-like waves, the potential is continuous. In general, there are two limit surfaces (x is multi-valued), but the regions of validity of (13) and (14) overlap and a shock (possibly of zero strength) must finally be inserted so as to satisfy the appropriate shock conditions (3) and (4). This process has been carried out in detail by Lighthill (1949) in the course of second-order conical solution near the Mach cone.

The velocity components of a flow over a flat-plate delta wing are conical in nature. The flow field is subdivided into three regions: a region interior to the vertex-centred Mach cone and two regions exterior to the Mach cone. These regions are shown in figure 1. The usual linear solution is, in region 1,

$$\begin{aligned}\varphi = -\frac{\alpha}{\pi}\Bigg\{&\frac{(x-Bny)}{B(1-n^2)^{\frac{1}{2}}}\cos^{-1}\frac{xn-By}{[(x-Bny)^2-B^2z^2(1-n^2)]^{\frac{1}{2}}}\\ &+\frac{(x+Bny)}{B(1-n^2)^{\frac{1}{2}}}\cos^{-1}\frac{xn+By}{[(x+Bny)^2-B^2z^2(1-n^2)]^{\frac{1}{2}}}\\ &-z\left[\pi+\tan^{-1}\frac{xy-Bn(y^2+z^2)}{z[x^2-B^2(y^2+z^2)]^{\frac{1}{2}}}-\tan^{-1}\frac{xy+Bn(y^2+z^2)}{z[x^2-B^2(y^2+z^2)]^{\frac{1}{2}}}\right]\Bigg\},\end{aligned} \tag{15}$$

where $n = k/B < 1$ and k is the tangent of the sweepback angle; in region 2,

$$\varphi = [-\alpha/B(1-n^2)^{\frac{1}{2}}]\,[x+Bny-Bz(1-n^2)^{\frac{1}{2}}]; \tag{16}$$

and in region 3, the result of replacing y by $-y$ in (16).

Since regions 2 and 3 are regions of constant velocity, the second-order solution may be arrived at conveniently by using an oblique transformation in conjunction with Van Dyke's (1952) solution for two-dimensional flows. The result is, in region 2,

$$f = [\alpha^2/B^2(1-n^2)]\,[x+Bny-Bz(1-n^2)^{\frac{1}{2}}]-[M^2N\alpha^2/2B^2(1-n^2)]\,[x+Bny]; \tag{17}$$

and in region 3, the result of replacing y by $-y$ in (17). The method of reverse flow will now be used to obtain the second-order solution interior to the Mach cone.

3. Generation of integral equation for second-approximation to surface flow using reverse-flow methods†

The flow over a given body can be connected with the flow over a related body for which the velocity at infinity is equal in magnitude but opposite in direction. The two free-stream Mach numbers and densities must also be equal. The equations that govern the first- or second-order forward and reverse flow can be written as

$$\nabla . \mathbf{W} = Q, \quad \nabla \times \mathbf{V} = 0, \quad \mathbf{V} \equiv u\mathbf{i}+v\mathbf{j}+w\mathbf{k}, \quad \mathbf{W} \equiv -B^2u\mathbf{i}+v\mathbf{j}+w\mathbf{k}. \tag{18}$$

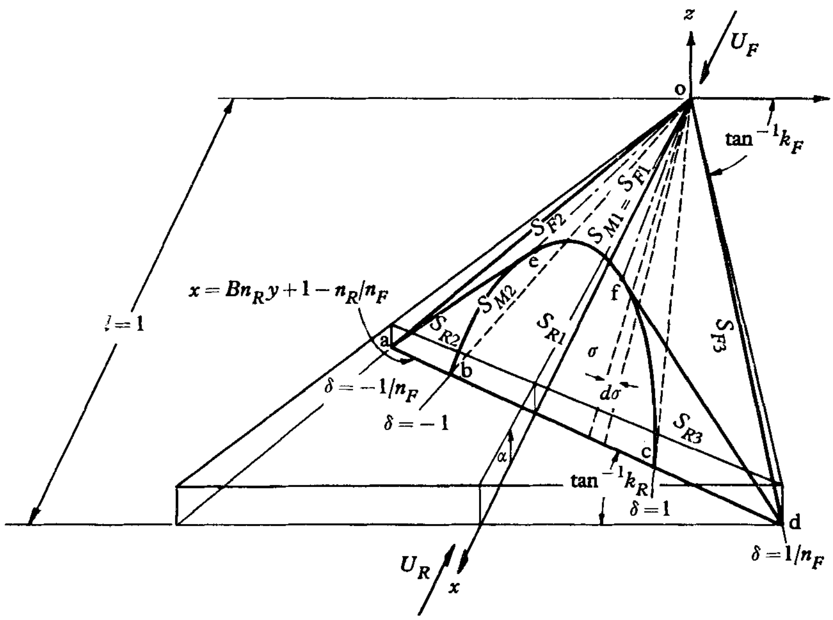

FIGURE 1. Envelopes of disturbance due to forward and reverse flow over delta wing.

The relevant volume-surface integral identity connecting two fields distinguished by F and R is (Ward 1955)

$$\int_S (\mathbf{V}_F\mathbf{W}_R.\mathbf{n}+\mathbf{V}_R\mathbf{W}_F.\mathbf{n}-\mathbf{V}_F.\mathbf{W}_R\mathbf{n})\,dS = -\int_T [\mathbf{V}_F\nabla.\mathbf{W}_R + \mathbf{V}_R\nabla.\mathbf{W}_F-\mathbf{W}_R\times(\nabla\times\mathbf{V}_F)-\mathbf{W}_F\times(\nabla\times\mathbf{V}_R)]\,dT, \tag{19}$$

where T is the region interior to a closed surface S with inward normal $\mathbf{n}$. Let $\mathbf{V}_F = \nabla(\varphi+f)_F$ be the non-dimensionalized second-order perturbation velocity vector in the forward flow and let $\mathbf{V}_R = \nabla\varphi_R$ be the non-dimensionalized (with respect to $U_R = -U_F$) velocity of the reverse flow. With the flow supersonic,

† When recapitulating necessary parts of the theory given by Clarke (1963), we use the occasion to offer alternate arguments when alternate arguments exist.

we apply (19) to the volume bounded by a closed surface S that is made up of the envelope S_F of disturbances in the forward flow, the envelope S_R of disturbances in the reverse flow, and the common reference surface σ. These surfaces are shown in figure 1.

The reverse flow is chosen to satisfy $\nabla . \mathbf{W}_R = 0$ and $\nabla \times \mathbf{V}_R = 0$ subject to the boundary condition (10a). The choice of the related body in the reverse flow is arbitrary but its reference surface must be coplanar with the reference surface of the body in the forward flow. Success with the method of reverse flow depends upon the skill with which the reverse-flow field is selected. To calculate the second-order surface pressure on the flat-plate delta wing, the reverse-flow wing is chosen here to have a constant angle of attack α_R and a leading edge that is swept at an unspecified angle $\tan^{-1} k_R$, as shown in figure 1. Although α_R is depicted in figure 1 as equal to α, it is important to note that α_R is an unrestricted parameter and no order estimate will be attached to it. The reverse-flow field has been chosen to satisfy the equations of linear theory as a convenient artifice, but any subsequent approximations made therein in accord with that theory will lead to error (see Clarke 1963). Since the edge of the wing in the reverse flow is supersonic, we have

$$\varphi_{Rx} = a_R/B(1-n_R^2)^{\frac{1}{2}}, \quad \varphi_{Ry} = -\alpha_R n_R/(1-n_R^2)^{\frac{1}{2}}, \quad \varphi_{Rz} = +\alpha_R, \; n_R = k_R/B. \tag{20}$$

We note that S_R is the plane

$$x - Bn_R y + B(1-n_R^2)^{\frac{1}{2}} z = (1 - n_R/n_F), \tag{21}$$

and the conical element of area on σ is

$$d\sigma = \frac{(1-n_R/n_F)^2}{2B(1-n_R\delta)^2} d\delta, \quad \delta \equiv By/x. \tag{22}$$

The specification of the reverse flow and its associated geometry is now complete.

Upstream of S_F, $\mathbf{V}_F \equiv 0$ and upstream of S_R, $\mathbf{V}_R \equiv 0$. Equation (19) should be applied only on either side of a surface of discontinuity in the vectors themselves. According to Ward (1955), sufficient conditions for the continuity of the integrand of S in (19) across a surface of discontinuity are

$$\Delta(\mathbf{n} \times \mathbf{V}) = 0, \quad \Delta(\mathbf{n} . \mathbf{W}) = 0, \tag{23a, b}$$

or, equivalently expressed in terms of the potential,

$$\Delta(\varphi + f) = 0, \quad \Delta[\boldsymbol{\tau} . \nabla(\varphi + f)] = 0, \quad \boldsymbol{\tau} \equiv -B^2 n_1 \mathbf{i} + n_2 \mathbf{j} + n_3 \mathbf{k}, \tag{24a, b}$$

where $\boldsymbol{\tau}$ is the so-called co-normal vector. The co-normal vector has the property that $\boldsymbol{\tau} . \mathbf{n} = 0$ for the undisturbed Mach surfaces S_F and S_R so $\boldsymbol{\tau} . \nabla$ is a directional derivative along these surfaces. If (24a) is satisfied on S_F and S_R, (24b) follows automatically: a sufficient condition for continuity of the integrand of S across S_F and S_R in (19) is that the potential be continuous. For vortex sheets the conditions given by (23) are independent. Any linearized solution, and in particular the reverse flow, satisfies (23) and no contributions occur from S over S_R in (19). The second-order potential is continuous near the Mach cone S_M and (23) is satisfied. However, the ordinary second-order potentials given by (17) are spuriously discontinuous across S_{F2} and S_{F3}, and the integrand of S over S_F is non-zero. But when we introduce the co-ordinate distortion (14) and determine x_1 to insure continuity of potential in accord with (3), no contributions to S from S_F will occur in the uniform second-order solution. Both the uniform and ordinary solutions should be identical away from S_F; this can only be true if there are no contributions to S from the integral over S_F even in the ordinary solution. This introduces a considerable simplification in applying the reverse-flow methods, since we can assume *a priori* that no contributions to S occur from the envelope of disturbance S_F.

After setting $\mathbf{n} = \mathbf{k}$ on σ and using (18), (20), and (22), we obtain from the scalar product of (19) and $\mathbf{U}_F$ the result

$$\int_{-1/n}^{1/n} \frac{(\varphi_x + f_x)}{(1-n_R\delta')^2} d\delta' + \frac{1}{B(1-n_R^2)^{\frac{1}{2}}} \int_{-1/n}^{1/n} \frac{(\varphi_z + f_z)}{(1-n_R\delta')^2} d\delta' = \frac{-2}{(1-n_R/n)^2(1-n_R^2)^{\frac{1}{2}}} \int_T Q\, dT, \tag{25}$$

where
$$Q = M^2[B^2(N-1)\varphi_x^2 + \varphi_y^2 + \varphi_z^2]_x, \tag{26}$$

and where the suffix F has been dropped. Since f_z is evaluated by equation (11), the only unknown quantity remaining is f_x. The corresponding statement for the linearized forward flow is obtained by suppressing the three second-order terms, and the result may be subtracted from (25). Two independent statements result, one for the known quantity φ_x and the other for the desired unknown quantity f_x. This unknown function of the auxiliary variable δ' can be considered to be governed by a singular integral equation whose current variable is the free parameter n_R. Its inversion for f_x allows the second-order surface pressure to be calculated, since the remaining terms in (12) may be calculated from the known linearized forward flow.

Before proceeding with a discussion of the integral equation (25), we introduce the partial particular integral $M^2\varphi\varphi_x$ of (8) given by Van Dyke (1952). This simplifies the volume integration since it accounts for all terms in Q except the term $M^2NB^2(\varphi_x^2)_x$. The integral identity (19) holds for any two vector fields mutually distinguished by F and R. With $\mathbf{V}_R$ the previous reverse-flow but $\mathbf{V}_F$ regarded as a fictitious flow defined by $\mathbf{V}_F = \nabla(M^2\varphi\varphi_x)$, we have

$$\int_{-1/n}^{1/n} \frac{M^2(\varphi\varphi_x)_x}{(1-n_R\delta')^2} d\delta' + \frac{1}{B(1-n_R^2)^{\frac{1}{2}}} \int_{-1/n}^{1/n} \frac{M^2(\varphi\varphi_x)_z}{(1-n_R\delta')^2} d\delta' = \frac{-2M^2}{(1-n_R/n)^2(1-n_R^2)^{\frac{1}{2}}} \int_T (-B^2\varphi_x^2 + \varphi_y^2 + \varphi_z^2)_x\, dT. \tag{27}$$

No contributions occur from S_F since $M^2(\varphi\varphi_x)$ is continuous there. Further, the term $M^2NB^2(\varphi_x^2)_x$ can be integrated with respect to x. Subtracting the linear result, using (27) and letting $b = 1/n_R$, we have the following two results

$$\int_{-1/n}^{1/n} \frac{\varphi_x}{(b-\delta')^2} d\delta' = -\frac{b}{B(b^2-1)^{\frac{1}{2}}} \int_{-1/n}^{1/n} \frac{\varphi_z}{(b-\delta')^2} d\delta', \tag{28}$$

$$\int_{-1/n}^{1/n} \frac{[f_x - M^2(\varphi\varphi_x)_x]}{(b-\delta')^2} d\delta' = -\frac{b}{B(b^2-1)^{\frac{1}{2}}} \int_{-1/n}^{1/n} \frac{[f_z - M^2(\varphi\varphi_x)_z]}{(b-\delta')^2} d\delta' + I(b), \tag{29}$$

where
$$I(b) = \frac{-2B^2M^2Nb}{(b-1/n)^2(b^2-1)^{\frac{1}{2}}}\left[\int_{S_R}\varphi_x^2\,dy\,dz - \int_{S_F}\varphi_x^2\,dy\,dz\right]. \tag{30}$$

We may now compare the integral equations (28) and (29) given by the method of reverse-flow with the more usual approaches. Since a complete particular integral for (8) is not known, the standard approach in solving second-order problems is to introduce the singular kernel

$$1/R_B \equiv 1/[(x-\xi)^2 - B^2(y-\eta)^2 - B^2(z-\zeta)^2]^{\frac{1}{2}}$$

and use one of Green's identities to derive a formal solution (Ward 1955). This solution contains a volume integration of the term Q/R_B that is virtually impossible to evaluate at the current point because Q is so complicated. The reverse-flow result contains a simpler kernel than the singular kernel $1/R_B$ in the volume integral and Q can always be integrated with respect to x, leaving only surface integrals. Also, the properties of the integral equation are well known and an explicit inversion can be presented. These simplifications achieved make the problem tractable.

4. Inversion of integral equation

After integration of (28) and (29) with respect to b from infinity to b and a change of the integration order on the left, both integral equations assume the form

$$\int_{-1/n}^{1/n}\frac{g(\delta')}{(\delta'-b)}\,d\delta' = H(b). \tag{31}$$

If b is continued analytically off the real axis, (31) becomes a singular integral equation for which a suitable inversion is available in the books of Widder (1940) or Tricomi (1957).

It is useful to consider (31) as an integral transform in which $b = \delta + i\eta$ is a complex variable. The transformation changes a real function g into a function H which is of the same class as g, namely L_p ($p > 1$). This class includes all flows in which the aerodynamic forces are finite. According to Widder (or Tricomi) (31) may be inverted by the complex formula

$$\tfrac{1}{2}[g(\delta+0) + g(\delta-0)] = \lim_{\eta\to 0}\frac{1}{2\pi i}[H(\delta+i\eta) - H(\delta-i\eta)]. \tag{32}$$

Substitution of $f_x - M^2(\varphi\varphi_x)_x$ for g gives formally the second-order solution, but a difficulty occurs in performing the required limiting procedure on H, or more specifically, on that part of H that depends upon I. It will be shown that the integral I becomes a singular integral and the limit $\eta \to 0$ cannot be performed without further manipulation. The essential step before applying (32) is to convert the surface integral I into a line integral; this can be accomplished by using the properties and transformations considered in conical flow theory.

5. Generation of another partial particular integral

Only the integration of φ_x^2 over S_{R1} presents problems in I, since φ_x is constant over S_{R2}, S_{R3} and S_F. In the interior of the Mach cone any component of the linearized velocity vector is the real part of an analytic function. For example, φ_x for the flat-plate delta wing can be written

$$\varphi_x = \frac{-\alpha}{B\pi(1-n^2)^{\frac{1}{2}}} \operatorname{Re}\left(\cos^{-1}\frac{n-t}{1-nt}+\cos^{-1}\frac{n+t}{1+nt}\right), \tag{33}$$

where

$$2/t = \xi+1/\xi, \quad \xi = [1-(1-\delta^2-\beta^2)^{\frac{1}{2}}]\,e^{i\theta}/(\delta^2+\beta^2)^{\frac{1}{2}},$$
$$\beta \equiv Bz/x, \quad \theta = \arg(\delta+i\beta). \tag{34}$$

Under the ξ-transformation, the interior of the Mach cone becomes the interior of the unit circle $|\xi| \leqslant 1$. The wing geometry is undistorted and an area element $dS_{R1}(y,z)$ is related to an area element $dS(\xi,\bar{\xi})$ in the ξ-plane by

$$dS_{R1}(y,z) = \frac{4(b-1/n)^2(1-\xi\bar{\xi})}{B^2b^2(\xi-e^{-is})^3(\bar{\xi}-e^{is})^3}, \quad e^{is} = \frac{1+i(b^2-1)^{\frac{1}{2}}}{b}, \tag{35}$$

where $\bar{\xi}$ is the complex conjugate of ξ. Extensive discussions of the properties of conical flows may be found in Lagerstrom (1950) and Ward (1955). Conversion from a surface integral to a line integral is facilitated by the following complex forms of Stokes's theorem

$$\int_S P_{\bar{\xi}}\,dS = \frac{1}{2i}\int_C P(\xi,\bar{\xi})\,d\xi, \quad \int_S P_{\xi}\,dS = -\frac{1}{2i}\int_C P(\xi,\bar{\xi})\,d\bar{\xi}. \tag{36}$$

This gives

$$I_{SR1} = \frac{4M^2N}{ib(b^2-1)^{\frac{1}{2}}}\left(\int_C \bar{P}\,d\bar{\xi}-P\,d\xi\right), \tag{37}$$

where

$$\begin{aligned}P(\xi,\bar{\xi}) = {} & \frac{\bar{\xi}\varphi_x^2}{2(\xi-e^{-is})^2(\bar{\xi}-e^{is})^2}+\frac{\xi\varphi_x\varphi_{x\xi}}{(\xi-e^{-is})^2(\bar{\xi}-e^{is})} \\ & +\frac{\bar{\xi}\varphi_{x\xi}\varphi_{x\bar{\xi}}}{(\xi-e^{-is})(\bar{\xi}-e^{is})}+\varphi_{x\xi}\int^{\bar{\xi}}\frac{(\bar{\xi}\varphi_{x\xi})_{\bar{\xi}}}{(1-\xi\bar{\xi})}\,d\bar{\xi} \\ & +\frac{e^{-is}}{(\xi-e^{-is})}\left[G(\xi)+(\xi\varphi_{x\xi})_{\xi}\int^{\bar{\xi}}\frac{\varphi_{x\bar{\xi}}}{(1-\xi\bar{\xi})}\,d\bar{\xi}+\varphi_{x\xi}\int^{\xi}\frac{(\bar{\xi}\varphi_{x\bar{\xi}})_{\bar{\xi}}}{(1-\xi\bar{\xi})}\,d\bar{\xi}\right],\end{aligned} \tag{38}$$

and $\bar{P}$ is found by replacing i by $-i$ in (38). The function G is discussed below. The closed contour C is composed of the upper semi-circle $\xi\bar{\xi} = 1$ and a straight line just above $\xi = \bar{\xi}$. Since φ_x is a constant exterior to the Mach cone, we integrate with respect to z and the total contribution to I from S_{R2}, S_{R3} and S_F gives a line integral around ofco and obeo in figure 1. Using Stokes theorem, we have the surface integral

$$I_{SR2}+I_{SR3}-I_{SF} = -\frac{2M^2NB^2b}{(b-1/n)^2(b^2-1)^{\frac{1}{2}}}\int_{S_{M2}+S_{M3}} \varphi_x^2\,dy\,dz. \tag{39}$$

Consider again I_{SR1}. By regarding the integrand of that part of I_{SR1} along cfeb as a function of y and z and by adding and subtracting line integrals along bo and oc, we then get line integrals along ocfebo and along obco. Through use of Stokes's theorem, it can be shown that the line integral along ocfebo cancels

(39). The surviving part of I now consists of a line integral around obco and, after integration by parts and a little manipulation, we have

$$I(b) = M^2N\int_{-1}^{1} \frac{\{\delta'[1+(1-\delta'^2)^{\frac{1}{2}}]\,\varphi_x\varphi_{x\delta'}-2F\}}{(\delta'-b)^2}\,d\delta', \tag{40}$$

where

$$F(\xi,\bar{\xi}=\xi) = \int^{\xi} G(\xi)\,d\xi + \int^{\bar{\xi}=\xi} G(\bar{\xi})\,d\bar{\xi} + \int^{\xi} d\xi \int^{\bar{\xi}=\xi} d\bar{\xi}(1-\xi\bar{\xi})^{-1}(\xi\varphi_{x\xi})_{\xi}\,\varphi_{x\bar{\xi}}$$
$$+\int^{\xi} d\xi \int^{\bar{\xi}=\xi} d\bar{\xi}(1-\xi\bar{\xi})^{-1}(\bar{\xi}\varphi_{x\bar{\xi}})_{\bar{\xi}}\,\varphi_{x\xi}. \tag{41}$$

In arriving at (40), we have assumed that $F(1) = F(-1)$ can be taken as zero. This will be verified.

The function G appearing in (38) and (41) occurs because, in integrating by parts with respect to one variable in (38), one may add any function of the other variable. After letting $b = \delta + i\eta$ in (37) and taking the limit $\eta \to 0$ under the integral sign, we obtain a singular line integral whenever the real part of b is less than one. Upon integration around C, a residue occurs at $b = \delta$ and G must be retained. For all other b, G may be considered zero. Thus, the character of I under the limit required by the inversion (32) changes from an analytic function of b into a singular integral with I determined only to an arbitrary analytic function.

By inserting (40) into (29), we see that the term in brackets in (40) has the same significance as the previously introduced partial particular integral $M^2\varphi\varphi_x$; let it be denoted by f_x^N. It may also be shown from (37) that f_z^N is zero on $z = 0$ since I_{SR1} does not contain a term of the form $b/(b^2-1)^{\frac{1}{2}}$. The integral I therefore supplies all the information required here about f^N, which can be properly interpreted as a partial particular integral of (8) itself. Equation (29) is now in the form (28). By analogy therewith, $[f - M^2\varphi\varphi_x - f^N]$ is identified with the complementary solution of (8). The inversion (32) may now be applied to (29) without difficulty.

To determine G we require a field argument. Strictly speaking, we have determined this partial particular integral only on the wing surface but, because ξ, $\bar{\xi}$ are independent variables, the form of the particular integral valid over the entire Mach cone can easily be found. After some rearrangement we have from (41)

$$f_x^N = -M^2N\Big\{(x\varphi_x^2/2)_x + G(\xi) + G(\bar{\xi})$$
$$+\int^{\xi} d\xi \int^{\bar{\xi}} d\bar{\xi}[(1+\xi\bar{\xi})(1-\xi\bar{\xi})^{-1}(\xi\varphi_x\varphi_{x\xi}+\bar{\xi}\varphi_x\varphi_{x\bar{\xi}})_{\bar{\xi}\xi} - \varphi_{x\bar{\xi}}\varphi_{x\xi}]\Big\}. \tag{42}$$

This agrees with a partial particular integral for f_x^N previously given by Moore (1950). From the nature of solutions to Laplace's and Poisson's equations, the most general form of a particular integral should contain terms of the type $[g(\xi) \pm g(\bar{\xi})]^2$ corresponding to such products of the linearized solution as φ_x^2, $\varphi_x\varphi_y$, etc. After the integrations in (42) are performed, we have only the product terms $g(\xi)\,g(\bar{\xi})$, but it is a simple matter to construct solutions of the form

$[g(\xi) \pm g(\bar{\xi})]^2$ such that f_x^N be real. The ambiguity is now reduced to a plus or minus sign and this can be resolved by requiring the potential f^N to be continuous at the Mach cone, or by using the Lighthill results at the Mach cone. Either method is equivalent to the requirement that the contribution from the double integrals in (42) vanish at the Mach cone. In performing the integrations, care must be exercised to choose the proper branch lines since the functions are complex. On the reference surface and for $0 \leqslant \delta < 1$, we have

$$\begin{aligned} f_x^N(\delta, \beta = 0) = -\tfrac{1}{2}M^2N\Big\{&(x\varphi_x^2)_x + [\varphi_x - \varphi_x(1)]^2 \\ &+ (1-n^2)\left[\frac{\varphi_x - \varphi_x(1)}{2n} + \frac{\alpha}{B\pi}\,\frac{n\cos^{-1}\{(n-\delta)/(1-n\delta)\} - \pi n}{(1-n^2)^{\frac{3}{2}}} - \frac{\alpha}{B\pi}\,\frac{(1-\delta^2)^{\frac{1}{2}}}{(1-n^2)(1-n\delta)}\right]^2 \\ &+ (1-n^2)\left[\frac{\varphi_x(1) - \varphi_x}{2n} - \frac{\alpha}{B\pi}\,\frac{n\cos^{-1}\{(n+\delta)/(1+n\delta)\}}{(1-n^2)^{\frac{3}{2}}} + \frac{\alpha}{B\pi}\,\frac{(1-\delta^2)^{\frac{1}{2}}}{(1-n^2)(1+n\delta)}\right]^2\Big\}. \end{aligned} \tag{43}$$

Symmetry properties determine f_x^N for $-1 < \delta < 0$.

6. Solution and results

The inversion of the integral equation gives for the second-order solution on the reference surface over $0 \leqslant \delta < 1$ the result

$$\begin{aligned} f_x(\delta, \beta = 0) = {} & M^2(\varphi\varphi_x)_x + f_x^N - \frac{\alpha^2}{1-n^2}\operatorname{Re}\frac{1}{\pi i}\left[\ln\left\{\frac{\delta n - 1}{\delta n + 1}\right\} - n\ln\left\{\frac{1-\delta}{1+\delta}\right\}\right] \\ & + \alpha^2 \operatorname{Re}\frac{1}{\pi i}\left[\frac{n^2}{1-n^2}\left(\ln\left\{\frac{1-n\delta}{1+n\delta}\right\} - \frac{1}{n}\ln\left\{\frac{1-\delta}{1+\delta}\right\}\right) + \frac{2}{1-n^2\delta^2} - \ln(1-n^2\delta^2)\right]. \end{aligned} \tag{44}$$

The solution for f_x has square-root singularities at the Mach cone and the ordinary method of successive approximations fails. The solution is made uniformly valid by application of the rule given in § 2. To the ordinary second-order potential we add the term $x_1(x,y,z)\,\varphi_x$ and replace x by u, where u and x are related by (14). A completely equivalent method is to add $x_1(x,y,z)\,\varphi_{xx}$ to f_x and then replace x by u. We now determine x_1 to insure that f_x is finite at the revised Mach cone $u = B(y^2+z^2)^{\frac{1}{2}}$. With $r = (\delta^2+\beta^2)^{\frac{1}{2}}$, this gives

$$x_1/u = -M^2\varphi(r=1)/u + M^2N\varphi_x(r=1), \tag{45}$$

and the revised Mach cone becomes

$$r_M = 1 - x_1/x + O(\alpha^2) = 1 + M^2\varphi(r=1)/x - M^2N\varphi_x(r=1) + O(\alpha^2). \tag{46}$$

The solution f_x now exhibits a jump in velocity

$$\Delta f_x = -M^2N[\varphi_x(r=1-0) - \varphi_x(r=1+0)]\,\varphi_{xx}(r=1). \tag{47}$$

As an example of the procedure, we have calculated f_x for $M = 3{\cdot}0$, $\gamma = 1{\cdot}4$, $n = 0{\cdot}35$ and $\alpha = 4°$; the results are shown in figure 2. On the suction side of the wing, the phenomenon near the Mach cone is one of shock and f_x correctly approximates this behaviour. On the pressure side the phenomenon near the Mach cone is one of expansion, indicating that f_x is in error since expansion waves are continuous. The width of the expansion wave though is of order α^2 and the change across the wave is given correctly by (47). The solution near the Mach cone was first given by Lighthill (1949) who did not attempt, however, to solve the problem away from the Mach cone.

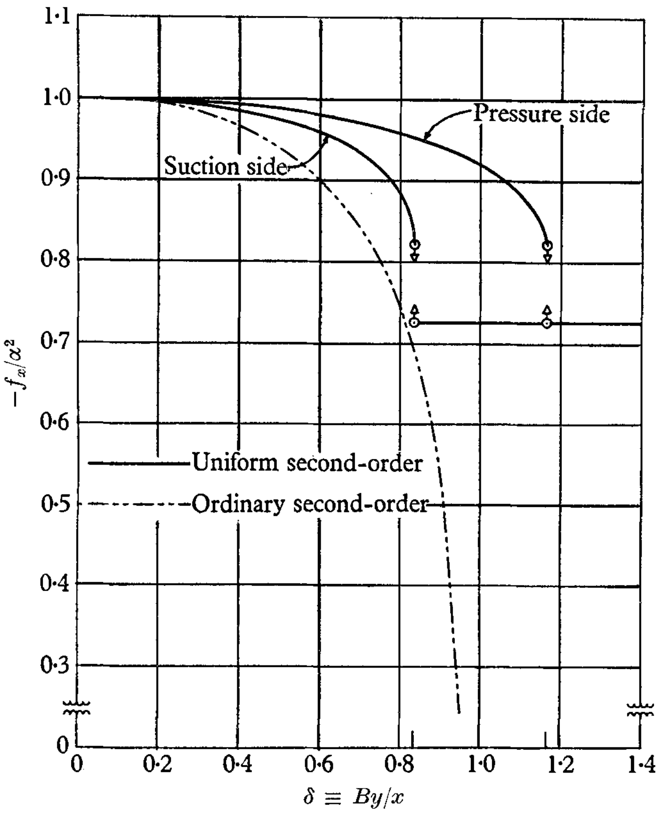

FIGURE 2. $-f_x/\alpha^2$ versus δ on reference surface of delta wing: $M = 3{\cdot}0$, $\gamma = 1{\cdot}4$, $n = 0{\cdot}35$, $\alpha = 4°$.

To find the shock strength we must first determine the limit surfaces, which have the property that the component of velocity normal to the surface is equal to the speed of sound. There are two limit surfaces (x_2 is multi-valued) and a shock, possibly of zero strength, must be inserted between them so as to satisfy (4). This has previously been done by Lighthill (1949) and his result is

$$\Delta c_p = 3M^2N[\varphi_x(r = 1-0) - \varphi_x(r = 1+0)]\,\varphi_{xx}(r = 1). \tag{48}$$

With neglect of the $O(\alpha^2)$ terms, the equation of the shock surface reduces to (46). For the previously specified conditions, the calculated variation of c_p with δ on the pressure and suction side of the wing is shown in figures 3 and 4, respectively. These results are compared with those of ordinary second-order theory, linear theory, and Fowell (1956). Further, these results exhibit the correct behaviour near $r = 1$ on both sides of the wing. For a reference to significant

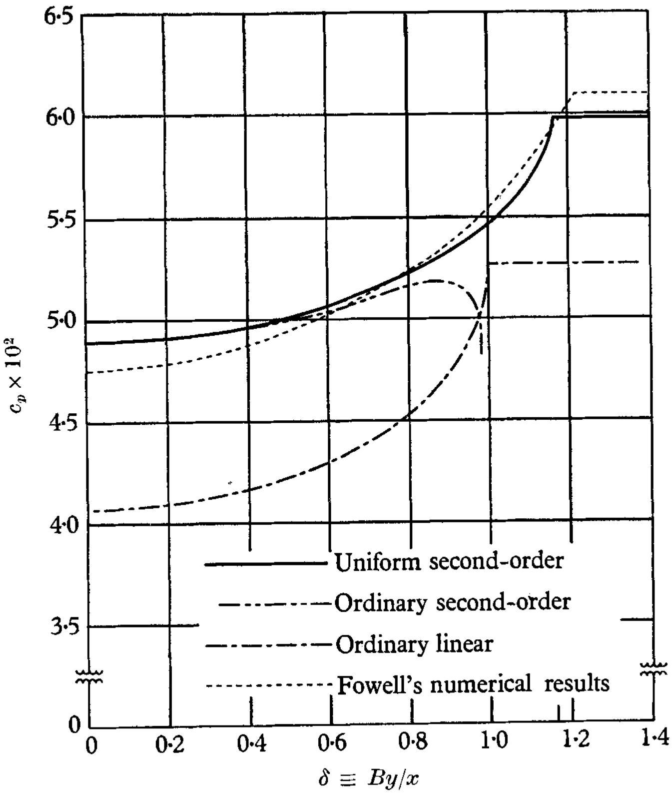

FIGURE 3. c_p versus δ on surface of delta wing (pressure side): $M = 3{\cdot}0$, $\gamma = 1{\cdot}4$, $n = 0{\cdot}35$, $\alpha = 4°$.

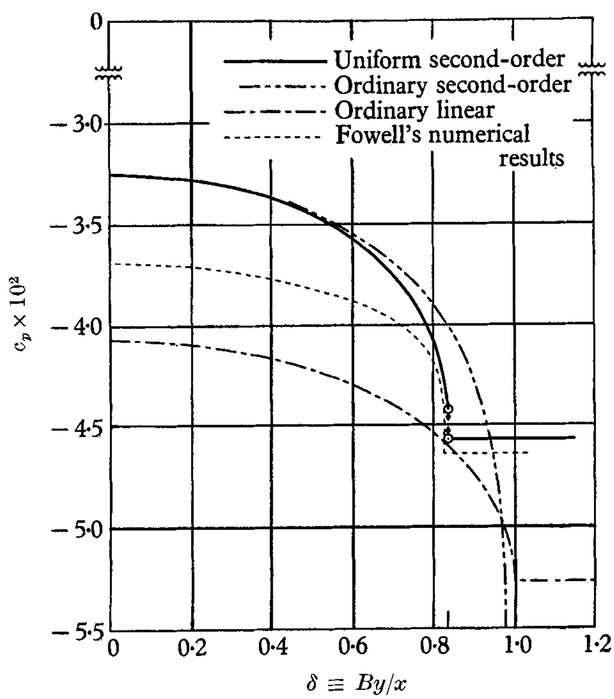

FIGURE 4. c_p versus δ on surface of delta wing (suction side): $M = 3{\cdot}0$, $\gamma = 1{\cdot}4$, $n = 0{\cdot}35$, $\alpha = 4°$.

recent Russian work on conical flow including commentary on the work of Fowell, the reader may refer to the interesting paper by Reyn (1960).

In conclusion we see that the reverse-flow methods have been successful in calculating a heretofore unsolved problem. In effect, the reverse-flow integral approach allows us to generate particular integrals to second-order problems; other bodies should then be amenable to these techniques. As significant as the analytic techniques given in this paper is the fact that the reverse-flow formulation is better suited for numerical techniques than the formulation based on Green's theorem, providing the singularities of the solution are understood. The given solution represents one of the few non-linear analytic wing solutions available, and can serve as a prototype for other three-dimensional problems that might require numerical techniques to solve. Also the solution presented in this paper can easily be generalized to include delta wings with arbitrary slope distribution, providing the flow remains conical.

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